**The Z-Transform**

**INTRODUCTION**

A sequence is a list of numbers, sequences can be finite, like (1, 2, 3, 4) or infinite, like (1, 2, 3, 4, 5 . . .). We are interested in infinite sequences. These all have the general form s with the  standing for the numbers in the sequence. We use the short hand:



The Z-transform is a transform for sequences. Just like the Laplace transform takes a function of *t* and replaces it with another function of an auxiliary variable *s*, well, the Z-transform takes a sequence and replaces it with a function of an auxiliary variable, *z*. The reason for doing this is that it makes difference equations easier to solve, again, this is very like what happens with the Laplace transform, where taking the Laplace transform makes it easier to solve differential equations.

The subject of solving recurrence relations (difference equations) arise in many areas such as **combinatory, probability theory, discrete time control theory, economics** etc. There are several powerful methods available to solve these equations such as, summing factors, generating functions, **Z transformations**, Operator methods etc. It has only been in the last few decades that interest in the **Z transform** has evolved, mostly due to the rapid development of integrated circuit technology and microprocessor architecture. **Z-transform** techniques have now become a major tool in electrical, computer, and communication engineering.

In [mathematics](https://en.wikipedia.org/wiki/Mathematics) and [signal processing](https://en.wikipedia.org/wiki/Signal_processing), the **Z-transform** converts a [discrete-time signal](https://en.wikipedia.org/wiki/Discrete-time_signal), which is a [sequence](https://en.wikipedia.org/wiki/Sequence) of [real](https://en.wikipedia.org/wiki/Real_number) or [complex numbers](https://en.wikipedia.org/wiki/Complex_number), into a complex [frequency domain](https://en.wikipedia.org/wiki/Frequency_domain) representation.

**Z-TRANSFORM**

The z-transform of a sequence is defined as

This equation is, in general, an infinite sum or infinite power series, with z being a complex variable. Sometimes it is useful to consider Eq.(1) as an operator that transforms a sequence into a function, and we will refer to the *z-transform operator* , defined as

With this interpretation, the z-transform operator is seen to transform the sequence into the function , where z is a continuous complex variable. The correspondence between a sequence and its z-transform is indicated by the notation

. (3)

The z-transform, as we have defined it in Eq.(1), is often referred to as the *two-sided or bilateral z-transform,* in contrast to the *one-sided or unilateral z-transform,* which is defined as

Clearly, the bilateral and unilateral transform are equivalent only if

For any given sequence, the set of values of *z* for which the z-transform converges is called the ***region of convergence*** (**ROC**). The z-transform is most useful when the infinite sum can be expressed in closed form, i.e., when it can be summed and expressed as a simple mathematical formula. Among the most important and useful z-transforms are those for which is a rational function inside the ROC, i.e.,

Where and are polynomials in . The values of for which are called the **zeros** of , and the values of for which is infinite are referred to as the **poles** of . The poles of for finite values of are the roots of the denominator polynomial.

**Discrete-Time Unit Step function**

The definition and the wave form of the Discrete-Time Unit Step function are shown as:

|  |  |
| --- | --- |
| where is an [integer](https://en.wikipedia.org/wiki/Integer). | C:\Users\ADMINI~1\AppData\Local\Temp\Rar$DI07.991\z-14.jpg |

**Properties and Theorems of z-Transform**

In the following discussion, denotes the z-transform of , and the *ROC* of is denoted by ; i.e.,

**Linearity property**

where *a*, *b* are arbitrary real or complex constants.

**Shift of**  in the Discrete-Time domain (Two-sided)

The quantity is an integer. If is positive, the original sequence is shifted right, and if is negative, is shifted left.

The derivation of this property follows directly from the z-transform expression in Eq.(1). If , the corresponding z-transform is

**Left Shift of**  in the Discrete-Time domain (One-sided),

**Proof:**

So, , and so on.

**Right Shift of**  in the Discrete-Time domain(one-sided)

This property is a generalization of the previous property, and allows use of non-zero values for

**Proof:**

So, , and so on.

**Multiplication by** in theDiscrete-Time domain

**Multiplication** byin theDiscrete-Time domain

**Multiplication** by andin theDiscrete-Time domain

By definition,

Differentiating one more time, we obtain the second pair.

**Convolution in the Discrete-Time domain.**

According to the convolution property,

**The z-Transform of simple Discrete-Time sequence**

**Example-1** The transform of the geometric sequence (**Right-sided exponential sequence).**

Consider the signal . The geometric sequence is defined as

Because it is nonzero only for , this is an example of right-sided sequence. From Eq.(1),

The ROC is the range of values of z for which , or equivalently, . Inside the ROC, the infinite series converges to

For is the unit step sequence with z-transform

The pole-zero plot and the ROC for Example-1 are shown in Fig.1, where a “o” denotes the zero and an “x” the pole.

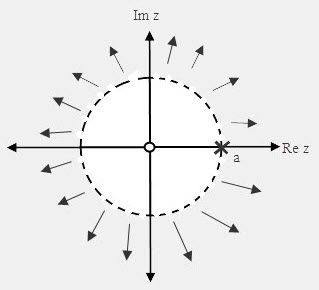


Fig. 1

**Example**-2 **Left-Sided Exponential sequence**

Consider the signal . Because it is nonzero only for , this is a *left-sided* sequence. Then

If , or equivalently, , the sum in Eq.(9) converges, and

(10)

Comparing Eq.(7) and Eq.(10), we see that the infinite sums are different, but the algebraic expressions for are identical. The z-transforms differ only in the ROC.

The pole-zero plot and the ROC for Example-2 are shown in Fig2.

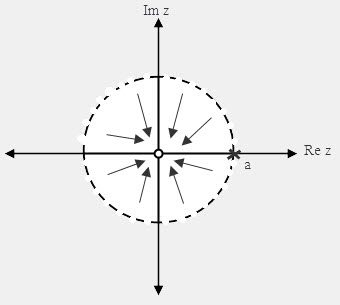


Fig. 2

**Example**-3 **Sum of Two Exponential sequences**

Consider a signal that is the sum of two real exponentials:

(11)

The pole-zero plot and the ROC for Example-3 are shown in Fig.3.

|  |  |  |
| --- | --- | --- |
| C:\Users\Administrator\Desktop\Figures of Z-trans\z-3.png | C:\Users\Administrator\Desktop\Figures of Z-trans\z-4.jpg | C:\Users\Administrator\Desktop\Figures of Z-trans\z-5.jpg |

Fig.3

The z-transform is then

For convergence of , both sums in Eq. (12) must converge, which requires both and or, equivalently and . Thus, the region of convergence is the region of overlap, .

**Example**-4 **Two-Sided Exponential sequence**

Consider the sequence

Note that this sequence grows exponentially as . Using the general result, we obtain

And

Thus by the linearity of the z-transform,

In this case, the ROC is the annular region . Note that the rational function in this example is identical to the rational function in Example-3 and 4, but the ROC is different in two cases.

The pole-zero plot and the ROC for Example-4 are shown in Fig.4.

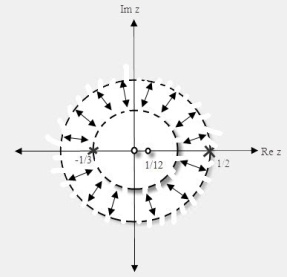


Fig.4

**Example**-5: Using the property of z-transform find , where .

Also find .

Solution: From the previous example we have

Therefore

.

Example-6 **Finite-Length Sequence**

Consider the signal

Then

The ROC is determined by the set of values of z for which

Since there are only finite number of nonzero terms, the sum will be finite as long as is finite, which in turn requires only that is finite, the ROC includes the entire z-plane, with the exception of the origin .

**Example-7** The Transform of the Discrete-Time Exponential Sequence

The discrete-time exponential sequence is defined as

Then

ROC: or,

**Example-8** The Transform of the Discrete-Time Cosine and Sine functions

Let

First, is expressed as

Then using Eq.(16) and the exponential multiplication property, we see that

From the linearity property, it follows that

**Example-**9 Similarly we can find z-transform for the function

**Kronecker delta function** or ***δ* function:** The Kronecker delta function, or function, is a [generalized function](https://en.wikipedia.org/wiki/Generalized_function), or [distribution](https://en.wikipedia.org/wiki/Distribution_%28mathematics%29), on the real number line that is zero everywhere except at zero.

|  |  |
| --- | --- |
| That is, | C:\Users\ADMINI~1\AppData\Local\Temp\Rar$DI00.065\z-15.jpg |

**Example-10**(a) **Delta function**

Find .

**Example-10**(b) **Delta function**

Find .

**Table-1** SOME COMMON z-TRANSFORM PAIRS

|  |  |  |
| --- | --- | --- |
| Sequence | Transform | ROC |
|  | 1 | All z |
|  |  | >1 |
|  |  | <1 |
|  |  | All z except 0 (if m>0) or (if m<0) |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |

**Exercise**- Prove all the z-transform of the sequences given in the table using definition.

PROPERTIES OF THE REGION OF CONVERGENCE FOR THE z-TRANSFORM

Property 1: The ROC is a ring or disk in the z-plane centered at the origin; i.e., 0 .

Property 2: The Fourier transform of converges absolutely if and only if the ROC of the z-transform of includes the unit circle.

Property 3: The ROC cannot contain any poles.

Property 4: If is a finite-duration sequence, i.e., a sequence that is zero except in a finite interval then the ROC is the entire z-plane, except possibly or

Property 5: If is a right-sided sequence, i.e., a sequence that is zero for the ROC extends outward from the outermost (i.e. largest magnitude) finite pole in to

Property 6: If is a left-sided sequence, i.e., a sequence that is zero for the ROC extends inward from the innermost (smallest magnitude) nonzero pole in to

Property 7: A two-sided sequence is an infinite-duration sequence that is neither right sided nor left sided. If is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.

**Stability, Causality, and the ROC**

Consider a system with impulse response for which the z-transform has the pole-zero plot shown in Fig.5. There are three possible ROC’s consistent with properties 1-7 that can be associated with this pole-zero plot. However, if we state in addition that the system is stable (or equivalently, that is absolutely summable and therefore has a Fourier transform), then the ROC must include the unit circle. Thus, stability of the system and properties 1-7 imply that the ROC is the region Note that as a consequence, is two sided, and therefore, the system is not causal.

If we state instead that the system is causal, and therefore that is right-sided, then property 5 would require that the ROC be the region Under this condition, the system would not be stable; i.e., for this specific pole-zero plot(Fig.5), there is no ROC that imply that the system is both stable and causal.

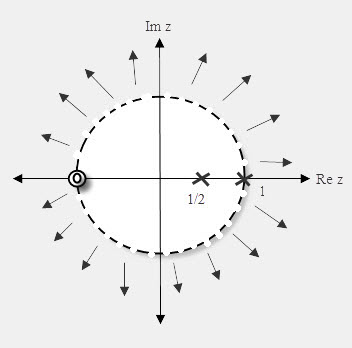
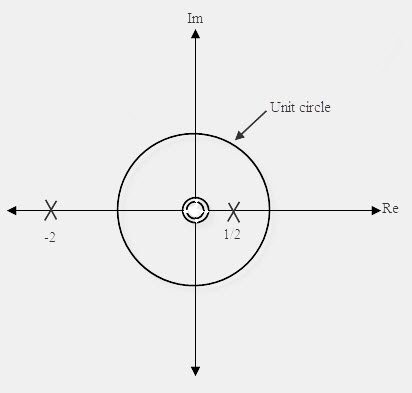


Fig. 5

**THE INVERSE z-TRANSFORM**

One of the important roles of the z-transform is in the analysis of discrete-time linear systems. Often, this analysis involves finding the z-transform of sequences and, after some manipulation of the algebraic expressions, finding the inverse z-transform. There are a number of formal and informal ways of determining the inverse z-transform from a given algebraic expression and associated ROC.

The inverse z-transform enables us to extract from . It can be found by any of the following methods:

1. Partial Fraction Expansion
2. The Inversion Integral
3. Long Division of polynomials

**Inspection method (using Table)**

Let us consider the sequence of the form , which is used quite frequently. We know

The inverse z-transform of

If the ROC associated with in Eq. (19) had been , we can write

1. **Partial fraction expansion**

To see how to obtain a partial fraction expansion, let us assume that is expressed as a ratio of polynomials in , i.e.,

(20)

Such z-transforms arise frequently in the study of linear time-invariant systems. To obtain the partial fraction expansion of , it is most convenient to note that could be expressed in the form

Where the ’s are the nonzero zeros of and the ’s are the nonzero poles of . If and the poles are all first order, then can be expressed as

**Example-11**

Consider the sequence with z-transform

|  |  |
| --- | --- |
|  | C:\Users\Administrator\Desktop\Figures of Z-trans\z-8.jpg  Fig.6 |

The pole-zero plot and the ROC for Example-11 are shown in Fig.6.

From the ROC, we see that is a right-sided sequence. Since the poles are both first order, can be expressed as

.

Therefore,

Since is right sided, the ROC for each term extends outward from the outermost pole. From Table and the linearity of the z-transform, it then follows that

If in Eq.(20), the complete partial fraction expansion would have the form

Example-2Consider the sequence with z-transform

|  |  |
| --- | --- |
|  | C:\Users\Administrator\Desktop\Figures of Z-trans\z-9.jpg  Fig.7 |

The pole-zero plot and the ROC for Example-8 are shown in Fig7.

From the ROC, it is clear that is a right-sided sequence. Since *M*=*N*=2 and the poles are all first order, can be represented as

The constant can be found by long division:

Thus can be expressed as

Now using Eq. (24), we obtain

Therefore,

From Table, we see that since the ROC is ,

Thus, from the linearity of the z-transform,

Example-3 **Finite-Length Sequence**

Suppose is given in the form

Partial fraction is not appropriate. However, by multiplying the factors, we can express as

Therefore, by inspection,

Equivalently,

|  |  |
| --- | --- |
|  | C:\Users\ADMINI~1\AppData\Local\Temp\z-13.jpg |

1. **The Inversion Integral**

The inversion integral states that

Where *C* is the closed curve that encloses all poles of the integrant, and by Cauchy’s residue theorem, this integral can be expressed as

Where represents a pole of and represents a residue at .

**Example**- Use the inversion integral method to find the Inverse z-transform of

Solution:

Multiplication of the numerator and denominator by yields

Now,

We are interested in the values of , that is , values of

For , we get

.

For , we get

.

For , there are no poles at , that is, the only poles are at and . Therefore

Now we can express

Example- Consider the z-transform

From the ROC, we identify this as corresponding to a right-sided sequence. We can rewrite in the form

Now can be expressed as

**Alternatively**, this problem can be solved by using the inversion integral method.

**Example**- Determine the inverse z-transform of

If

1. .

Solution: Using partial fraction,

1. In the case when , the signal is causal and both terms are causal terms. Taking inverse z-transform, we get
2. When the , the signal is anticausal. Thus both the terms result in anticausal components. Taking inverse z-transform, we get
3. In the case when is a ring, which implies that the signal is two-sided. Thus one of the terms corresponds to a causal signal and the other to an anticausal signal. Obviously, the given ROC is the overlapping of the regions and . Hence the pole provides the causal part and the pole the anticausal. Thus

The **convolution** property plays a particularly important role in the analysis of **LTI** (Linear Time-Invariant) systems. Specifically, as a consequence of this property, the z-transform of the output ( of an LTI system is the product of the z-transform of the input (i.e. ) and the z-transform of the system impulse response (i.e. ). The z-transform of the impulse response of an LTI system is typically referred to as the system function.

Thus we can write

We can find the discrete impulse response by taking inverse z-transform of the discrete transfer function / system function that is,

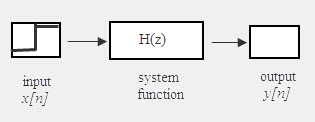


Table 2 SOME z-TRANSFORM PROPERTIES

|  |  |  |
| --- | --- | --- |
| Sequence | Transform | ROC |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  | , except for the possible addition or deletion of the origin or . |
|  |  |  |
|  |  | , except for the possible addition or deletion of the origin or . |
|  |  |  |

**Sample Exercises on Z-transform-8.1**

1. Prove the following z-transform pairs (using definition):

(a) (b) (c)

(d) (e) .

2. Determine the sequence with z-transform and hence sketch

3. Determine the inverse z-transform for the following.



(b) (c)

(d)

4. Use partial fraction expansion and the inversion integral method to find

(a) (b)

(c) (d)

(e)

5. If and , determine the values of

Answer: 2. 3(a). , 3(b). , 3(c), 3(d). 4(a). , 4(b). 4(c). , 4(d). , 4( e). , 5. .

**DIFFERENCE EQUATION**

Difference equations are the discrete equivalent of differential equations. The terminology is similar and the methods of solution have much in common with each other. Difference equations arise whenever an independent variable can have only discrete values. They are growing importance in engineering in view of their association with discrete time systems based on the microprocessor.

Example - The difference equation describing the input-output relationship of a discrete-time system with zero initial conditions, is

Compute:

1. The transfer function H(z)
2. The discrete-time impulse response
3. The response when the input is the discrete unit step function .

**Solution**

1. Taking z-transform of both sides of Eq.(1), we obtain

And thus

1. To obtain the discrete-time impulse response , we need to compute the inverse z-transform of Eq.(2).

.

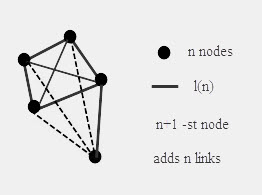
1. From , the transform , and using the result of part(a) we obtain:

.

**Application of the z-Transform to Connectivity**

**Example**: If we have *n* nodes connected to each other, how many total links do we have?

**Solution**:

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If a new node is added to the previous *n* nodes, exactly *n* new links are created. If we call the number of initial links *l*(*n*) then the number of links between *n*+1 nodes will be

.

This is a difference equation for *l* for which a closed form solution can be obtained with the z-transform. Let us define . Taking z-transform of the above equation, we get

There are zero links for zero nodes, thus . Therefore we have

which consists of a pole at of order 3. Taking inverse z-transform,

**Application of the z-Transform to Mathematics**

**Example :** The sequence 0, 1, 1, 2, 3, 5, 8, 13,… is said to form the Fibonacci numbers. Find the difference equation satisfied by them. Find an explicit formula for the nth Fibonacci number.

**Solution:**

It can be observed that sum of two consecutive numbers is the third number. Hence it satisfies the recurrence relation

, 

with the initial conditions

So,

Now taking *Z*-transforms, we have

Now,

Where and are the poles of and .

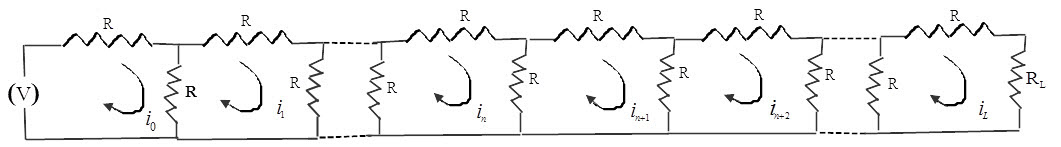
**Example:** Find a recurrence relation and initial conditions for the sequence.

**Solution:**

Finding the recurrence relation would be easier if we had some context for the problem (like the Tower of Hanoi, for example). But we have only the sequence. Remember, the recurrence relation tells, how to get from previous terms to future terms. What is going on here? We could look at the differences between terms: 4, 12, 36, 108, …. Notice that these are growing by a factor of 3. Is the original sequence as well? 1⋅3=3, 5⋅3=15, 17⋅3=51 and so on. It appears that we always end up with 2 less than the next term.

So  is our recurrence relation and the initial condition is .

**Ladder Network**

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It is required to find the current in the *n*-th loop for the ladder network shown in the Fig. Assume all the resistances except the load have the same value

The equation for the loop with current is

This equation is true for any except and (the beginning and the end loops). The equation with end conditions is sufficient to describe the network.

Applying z-transformation and dividing by , we obtain

Let us write the equation for the first loop

or,

**System of Difference Equations**

**Example:** Solve, by z-transformation, the simultaneous difference equations,

with

**Solution:** Taking z-transform of both the equations

Now using Cramer’s rule, we get

and

Hence, taking inverse z-trans

and v.

**Sample exercises on application of Z-transform-8.2**

1. Consider an LTI system with input and output that satisfies the difference equation with zero initial conditions

Compute

1. The transfer function .
2. The discrete-time impulse response .
3. The response when the input is the discrete unit step function

Ans: (a)=, (b) (c)

1. Find the impulse response of casual system having system function .

Ans: .

1. A discrete-time system is described by the difference equation

Where for .

1. Compute the transfer function .
2. Compute the impulse response
3. Compute the response when the input is .
4. Given the discrete function

.

Write the difference equation that relates the output to the input .

1. Find the recurrence relation and initial condition for the following sequences:
   1.  Ans: .
   2.  Ans: 
2. Use the one-sided z-transform to determine in the following cases:
3. .

Ans: .

Ans: .

Ans: .

1. .

Ans: .

1. .

Ans.

1. .

Ans: .

1. Assume that the population of a country in 2010 is 140 million and is growing at the rate of 2.5% a year.
2. Find a recurrence relation and initial condition for the population of the country *n* years after 2010?
3. Find an explicit formula for the population of the country *n* years after 2010.
4. Find the population of the country at the end of the year 2020.
5. Suppose that the number of bacteria in a colony doubles every hour.
6. Find a recurrence for the number of bacteria after *n* hours have elapsed.
7. If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?
8. A deposit of Tk. 1,00,000 is made in an investment fund at the beginning of a year. On the last day of each year two dividends are awarded. The first dividend is 20% of the amount in the account during that year. The second dividend is 28% of the amount in the account in the previous year.
9. Find a recurrence relation for , where  is the amount in the account after *n* years if no money has been withdrawn?
10. How much is in the account after *n* years if no money has been withdrawn?

**Compute z-transform and inverse z-transform using MATLAB**

Ex-1 Compute the z-transform of the function using MATLAB

>> syms n x

>> f=sin(n);

>> ztrans(f, n, x)

ans =

(x\*sin(1))/(x^2 - 2\*cos(1)\*x + 1)

Ex-2 Compute the z-transform of the function using MATLAB

>> syms a n x

>> f=a^n;

>> ztrans(f, x)

ans =

-x/(a - x)

Ex-3 Compute the inverse z-transform of the function using MATLAB

>> syms k x

>> F=2\*x/(x-2)^2;

>> iztrans(F, x, k)

ans =

2^k + 2^k\*(k - 1)

Ex-4 Compute the inverse z-transform of the function using MATLAB

>> syms z a n

>> F=exp(a/z);

>> iztrans(F)

ans =

a^n/factorial(n)

